q-Statistics on symmetric generalized binomial distributions

G. Ruiz López[a], C. Tsallis[b]

[a] Universidad Politénica de Madrid (Spain).
[b] Centro Brasileiro de Pesquisas Fisicas and
National Institute of Science and Technology for Complex Systems (Brazil);
Santa Fe Institute (USA) .

POLITÉCNICA

Physical scenario of a possible Large deviation Theory (LDT) generalization

a) Standard many-body Hamiltonian system in thermal equilibrium (T)

- BG weight: $e^{-\beta H_N} \sim e^{-\beta\,[H_N/N]N}, \quad \beta = 1/k_B T$

  (short-range + ergodic = *extensive* energy)

- LDT probability: $P(N) \sim e^{-r_1 N}, \quad r_1 \equiv$ rate function

  ($r_1 \sim$ BG relative entropy *per particle*)

b) *d*-dimensional classical system: 2-body interactions $V(r) \sim -1/r^{\alpha}$

- Long ranged $(0 \le \alpha/d \le 1)$:

$$e^z \to e_q^z \equiv [1+(1-q)z]^{\frac{1}{1-q}} \quad \left(e_1^z = e^z\right); \quad \ln_q z = \frac{z^{1-q}-1}{1-q} \quad \left(\ln_1 z = \ln z\right)$$

$$\beta H_N \to \tilde{\beta}\tilde{H}_N = \tilde{\beta}\left[\tilde{H}_N/N\right]N \quad \left(\tilde{\beta} = \beta\tilde{N}, \tilde{H}_N = H_N/\tilde{N}, \tilde{N} = \ln_{\alpha/d} N\right)$$

($\tilde{\beta}\tilde{H}_N/N \sim$ *intensive* variable)

- LDT probability:

$$P(N) \sim e^{-r_1 N} \to \boxed{P(N) \sim e_q^{-r_q N}, \quad r_q \equiv q\text{-rate function}}$$

[G. Ruiz & C. Tsallis, Phys. Lett .A **376** (2012) 2451-2454.]
[G. Ruiz & C. Tsallis, Phys. Lett. A **377** (2013) 491-495.]

Binomial distribution and LDT

Sequence of $N \in \mathbb{N}$ independent trials (two outcomes)

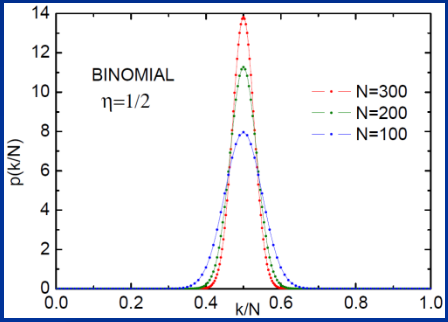
Probability of "win": $\eta \in [0,1]$; Probability of "loss": *(1−η)*

Number of "win" outcomes: $\overbrace{k=0, k=1, \dots, k=N}^{N+1}$

Number of outcomes containing *k* "wins": $C_{N,k} = \binom{N}{k} = \frac{N!}{k!(N-k)!}$

Probability of obtaining *k* "wins":

$$\boxed{p_k^{(N)}(\eta) = \frac{N!}{(N-k)!k!}\eta^k (1-\eta)^{N-k}}$$



Probability of the ratio of wins per trial $p(k/N) = N p_k^{(N)}(\eta)$

$$\left(\eta = \frac{1}{2}\right) \Rightarrow \boxed{P\left(N; \frac{k}{N} < x\right) \equiv \sum_{\left\{k:\, \frac{k}{N} < x\right\}} p_k^{(N)} = \sum_{\left\{k:\, \frac{k}{N} < x\right\}} \binom{N}{k} \frac{1}{2^N} \quad (0 \le x \le 1/2)}$$

Weak Law of large numbers:

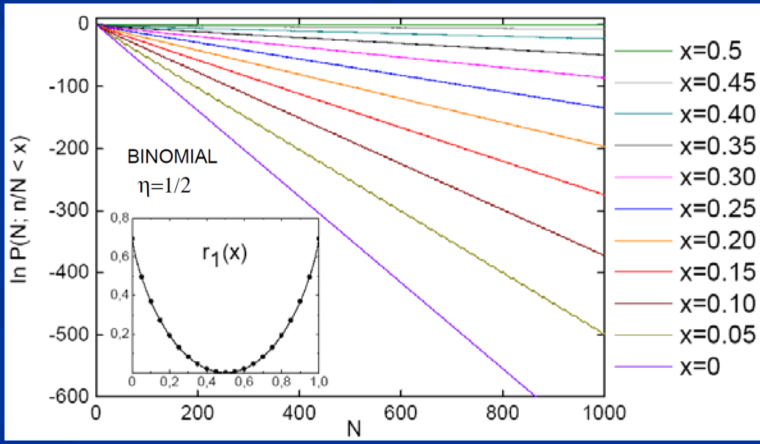
$$\lim_{N\to\infty} P\left(N; \left|\frac{k}{N} - \frac{1}{2}\right| > \varepsilon\right) = 0 \quad \forall \varepsilon > 0 \Leftrightarrow \lim_{N\to\infty} P\left(N; \frac{k}{N} < x\right) = 0 \quad \forall x \le \frac{1}{2}$$

Rate at which limit is attained:

$$\lim_{N\to\infty} \frac{1}{N} \ln P\left(N; \frac{k}{N} < x\right) = -\underbrace{\left[x \ln x + (1-x)\ln(1-x) + \ln 2\right]}_{r_1(x):\ \text{relative entropy}} \equiv -r_1(x)$$

$$\Downarrow$$

$$\boxed{P\left(N; \frac{k}{N} < x\right) \simeq e^{-N\, r_1(x)}}$$



Large Deviation Principle (r_1: rate function)

Generalized Binomial distribution

Sequence of $N \in \mathbb{N}$ nonnegative strictly increasing real numbers, $\chi = \{x_N\}_{N\in\mathbb{N}}$

$$p_k^{(N)}(\eta) = \frac{N!}{(N-k)!k!}\eta^k(1-\eta)^{N-k} \quad \Rightarrow \quad \boxed{\mathfrak{p}_k^{(N)}(\eta) = \frac{x_N!}{x_{N-k}!x_k!}q_k(\eta)q_{N-k}(1-\eta)}$$

where: $N! \quad \to \quad x_N! \equiv x_1x_2\ldots x_N, (x_0! = 1)$

$\eta^k \quad \to \quad q_k(\eta) \equiv$ polynomial of degree k

[H. Bergeron, E. M. F. Curado, and J. P. Gazeau, J. Math. Phys. 54, (2013) 123301]

Properties:

- $\mathfrak{p}_k^{(N)}(\eta)$ **preserves the symmetry win-loss: invariant under (*k* → *N-k*, *η* → *1-η*)**
- $\mathfrak{p}_k^{(N)}(\eta)$ **represent probabilities of *k* wins in a sequence of *N* correlated trials:**

$$\left.\begin{array}{l} \text{a)}\ \sum_{k=0}^{N}\mathfrak{p}_k^{(N)}(\eta) = 1 \quad \forall N \in \mathbb{N}, \forall \eta \in [0,1] \\ \text{b)}\ \mathfrak{p}_k^{(N)}(\eta) \geq 0 \quad \forall N, k \in \mathbb{N}, \forall \eta \in [0,1] \end{array}\right\} \Rightarrow \text{generating functions of } \mathfrak{p}_k^{(N)}(\eta)$$

Spetial case: *q*-exponential generating function

$$F(\eta\,;t) \equiv \sum_{n=0}^{\infty}\frac{q_n(\eta)}{x_n!}t^n;\ F(\eta\,;t)F(1-\eta\,;t) = (1-t/\alpha)^{-\alpha}, \alpha > 0 \quad \Rightarrow \quad \mathfrak{p}_k^{(N)}(\eta) = \frac{\binom{-\eta\alpha}{k}\binom{-(1-\eta)\alpha}{N-k}}{\binom{-\alpha}{N}}$$

Properties:

- **Fulfills the Leibniz triangle rule** $\Rightarrow$ S_{BG} extensive ($q_{\text{entropy}} = 1$)
- **Ordinary binomial limit (** $\alpha \to \infty$ **)**
- **Expectation value** $\langle k \rangle_N(\eta) = N\eta$ **; Variance** $(\sigma_k)_N^2(\eta,\alpha) = N^2\eta(1-\eta)\dfrac{1+\alpha/N}{1+\alpha}$

Ratio of wins per trial

A) Finite sequence of N nonnegative strictly increasing real numbers $\chi = \{x_k\}_{k=0}^{N}$

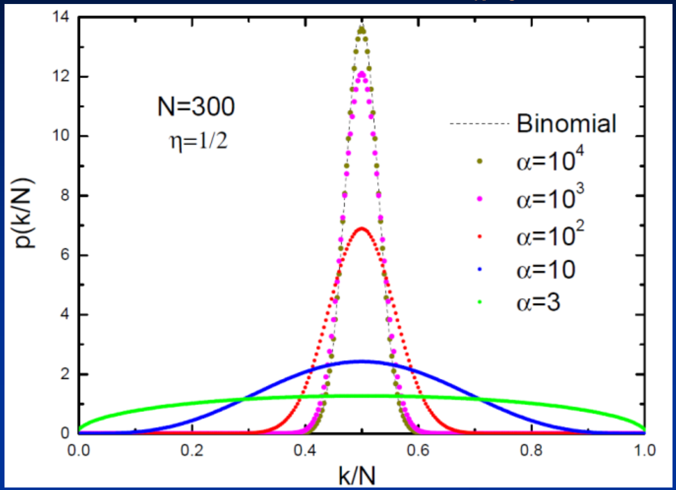
Number of "win" outcomes: $\overbrace{k=0, k=1, \ldots, k=N}^{N+1} \Rightarrow 0 \le k/N \le 1$

Probability of ratio of wins per trial k/N: $\boxed{p(k/N) = N\, \mathfrak{p}_k^{(N)}(\eta,\alpha)}$

$$\downarrow \alpha \to \infty$$

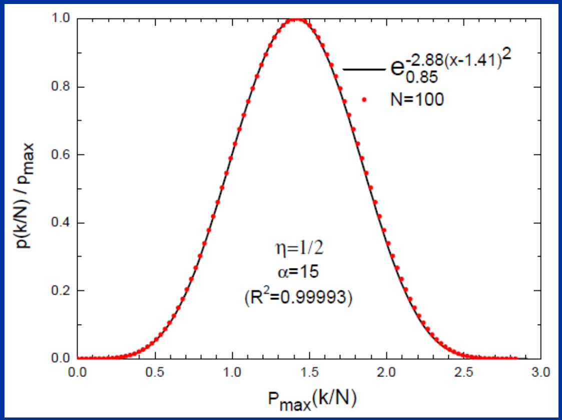
$$p(k/N) = N p_k^{(N)}(\eta)$$

Binomial distribution limit



Normalized distribution of $p(k/N;\eta=1/2,\alpha) = N\,\mathfrak{p}_k^{(N)}(\eta=1/2,\alpha)$

Numerical results strongly suggest that, $\forall \alpha, N$:

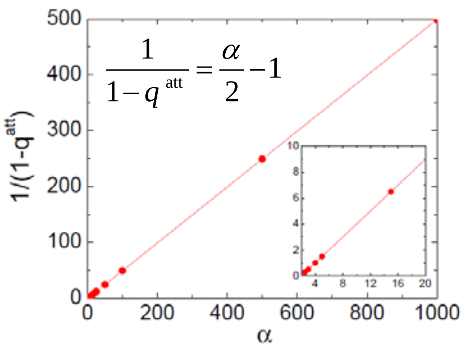


$$p(k/N) = p_{\max}\, e_q^{-\beta\left[p_{\max}\left(\frac{k}{N}-\frac{1}{2}\right)\right]^2}$$

where $e_q^{-\beta x^2} \equiv [1-\beta(1-q)x^2]^{1/(1-q)}$

Numerical values of q:

α	$N=100$	$N=1000$	$N=5000$
3	...	...	-1
4	...	...	0
5	...	...	$0.\overset{\frown}{3}$
15	0.847	0.85	
25	0.9	0.915	
50	0.95	0.955	
100	0.97	0.978	
500	0.987	0.995	



Normalized distribution evolves as q-Gaussians,
The value of q increases with N, α.

$$\boxed{q^{att}(\alpha) \equiv \lim_{N\to\infty} q(\alpha, N) = 1 - \frac{2}{\alpha-2}}$$

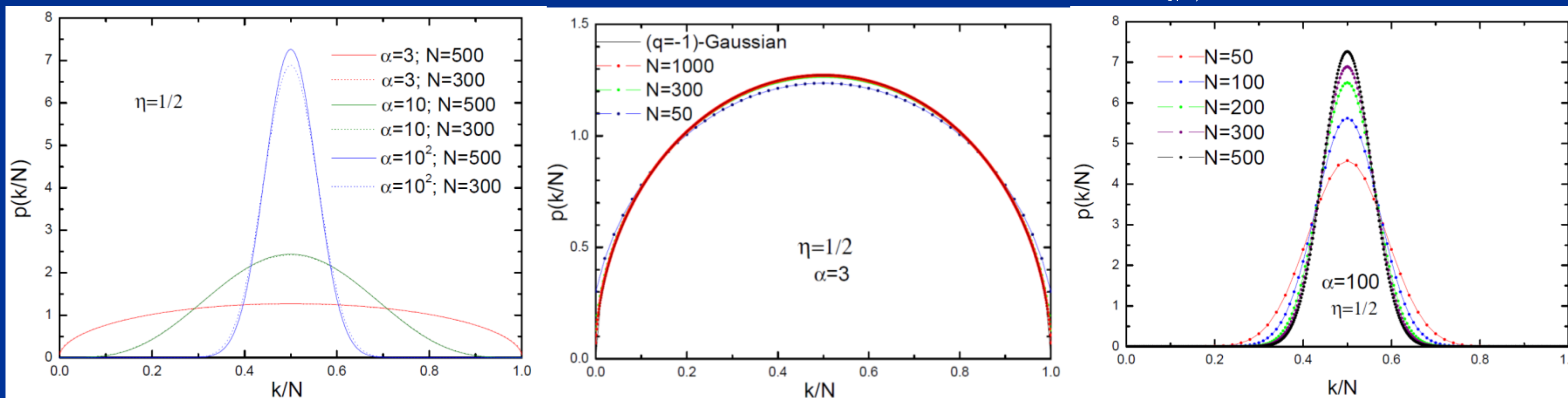
Ratio of wins per trial

B) Infinite sequence of N nonnegative strictly increassing real numbers $\chi = \{x_N\}_{N\in\mathbb{N}}$

Centered and scaled probabilities $\left(\frac{k-N/2}{N^{\gamma}}, N^{\gamma}\, \mathfrak{p}_k^{(N)}\right)$ colapses when $\gamma = 1 \Rightarrow$ **superdiffusion process** $(\gamma > 1/2)$

$\Downarrow$

$(q<1)$-Gaussian a**ttractors in probability space,** $G_q(k/N\,|\,\eta,\alpha) = \lim\limits_{N\to\infty} N\, \mathfrak{p}_k^{(N)}(\eta,\alpha)$



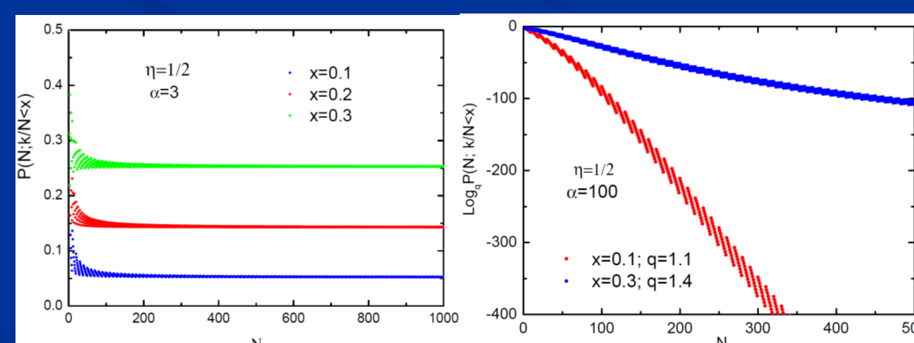
Probability of deviation from a ratio of wins per trial

$$P\left(N;\frac{k}{N}<x\,|\,\eta,\alpha\right) = \sum_{\left\{k:\frac{k}{N}<x\right\}} \mathfrak{p}_k^{(N)}(\eta,\alpha) \approx \int_{0\le x\le \frac{k}{N}} p(k/N)\,dx$$

$$\sqcap$$

$$\lim_{N\to\infty} N\,\mathfrak{p}_{k=N/2}^{(N)}(\eta,\alpha) = \text{cte} \neq \infty \;\; \forall \eta,\alpha$$

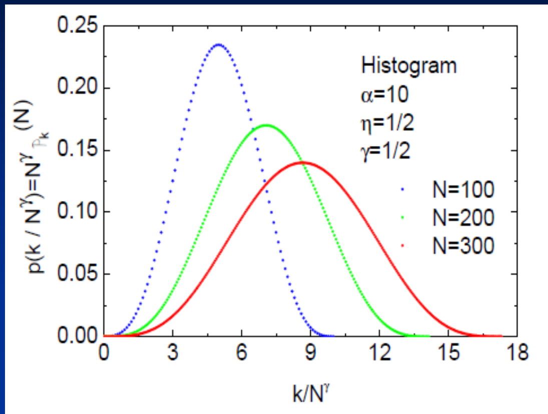
$$\Downarrow$$

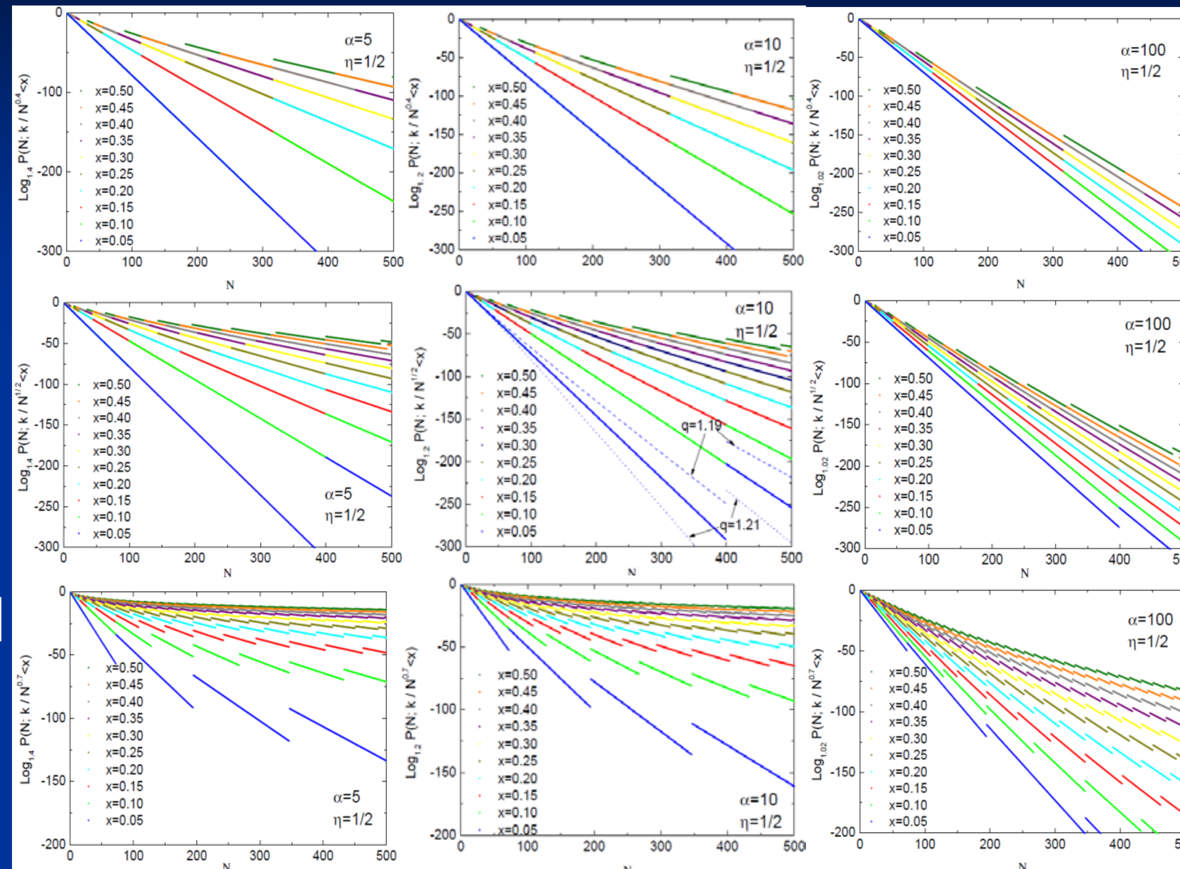


The Weak Law of large numbers is NOT satified:

$$\boxed{P\left(N;\left|\frac{k}{N}-\frac{1}{2}\right|>\varepsilon\,|\,\eta,\alpha\right) = 2\sum_{\left\{k:\frac{k}{N}<\left(x\equiv\frac{1}{2}-\varepsilon\right)\right\}} \mathfrak{p}_k^{(N)}(\eta,\alpha) \Rightarrow \lim_{N\to\infty} P\left(N;\left|\frac{k}{N}-\frac{1}{2}\right|>\varepsilon\,|\,\eta,\alpha\right) \neq 0 \quad \forall\varepsilon>0, \forall\eta,\alpha.}$$

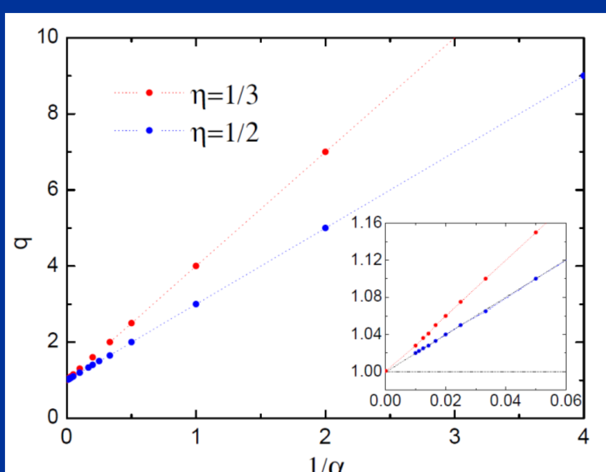
$P\,(N; \frac{k}{N^{\gamma}} < x \,|\, \eta, \alpha)$ distributions $(\gamma < 1)$





q-exponential decay:

$$\boxed{P\,(N; \frac{k}{N^{\gamma}} < x \,|\, \eta, \alpha) \sim e_q^{-r\,N}}$$



$q=q(\alpha,\gamma)$ dependence:

$$\boxed{q(\eta,\alpha,\gamma) = 1 + \frac{1}{\eta\alpha}} \qquad (2 < \alpha < \infty \Rightarrow\ 1 < q(\alpha)\big|_{\eta=1/2} < 2)$$

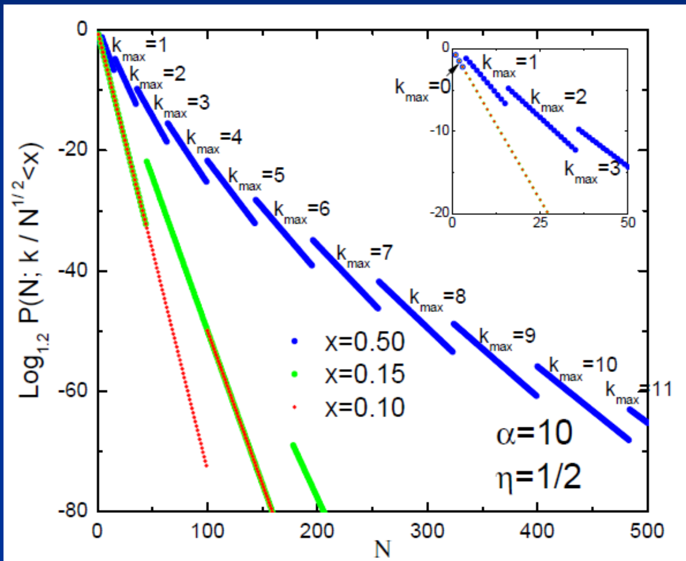
$$\Rightarrow \alpha\big|_{\eta=1/2} = \frac{2}{q(\alpha)\big|_{\eta=1/2} - 1} = 2 - \frac{2}{q^{att}(\alpha) - 1} \quad \Rightarrow \boxed{\frac{1}{q(\alpha)\big|_{\eta=1/2} - 1} + \frac{1}{q^{att}(\alpha) - 1} = 1}$$

Slopes dependence?

$$r = r(x, N, \gamma \,|\, \eta, \alpha)\ldots$$

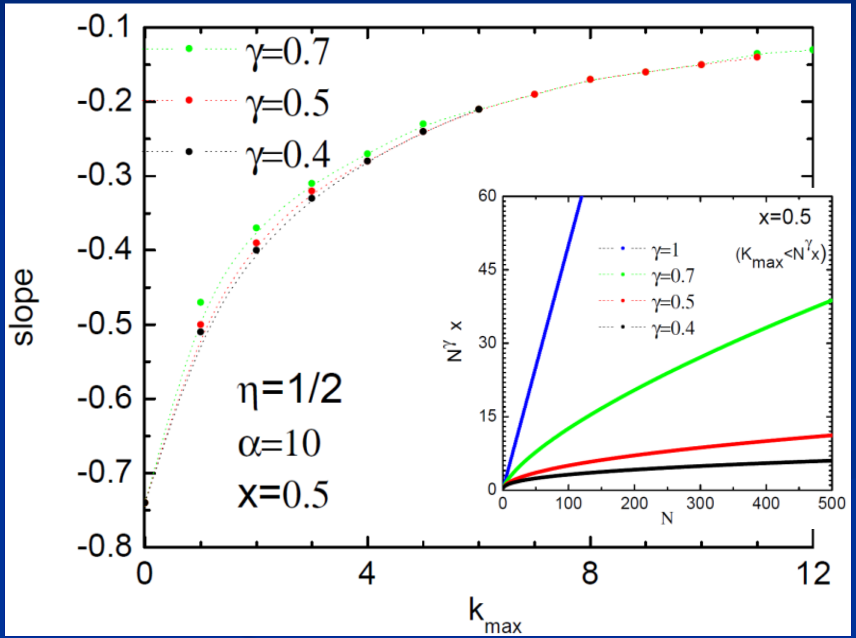
$P\left(N;\frac{k}{N^{\gamma}}<x\,|\,\eta,\alpha\right)$ distributions $(\gamma<1)$

$r=r(x=0.5,N,\gamma=0.5\,|\,\eta=0.5,\alpha=10):$



$$\left.\begin{array}{c} k<N^{\gamma}x \\ x=0{,}5 \\ (\gamma=1/2) \end{array}\right\} \Rightarrow k<\frac{\sqrt{N}}{2}:\left\{\begin{array}{l} 1\le N\le 3\Rightarrow k_{max}=0 \\ 4\le N\le 15\Rightarrow k_{max}=1 \\ 16\le N\le 35\Rightarrow k_{max}=2 \\ 36\le N\le 63\Rightarrow k_{max}=3 \\ 64\le N\le 99\Rightarrow k_{max}=4 \\ 100\le N\le 143\Rightarrow k_{max}=5 \\ 144\le N\le 195\Rightarrow k_{max}=6 \\ 196\le N\le 255\Rightarrow k_{max}=7 \\ 256\le N\le 323\Rightarrow k_{max}=8 \\ 324\le N\le 339\Rightarrow k_{max}=9 \\ 400\le N\le 483\Rightarrow k_{max}=10 \\ 484\le N\ldots\Rightarrow k_{max}=11 \end{array}\right.$$

$(\gamma\to 1):$



q-exponential decay:

$$P\left(N;\frac{k}{N^{\gamma}}<x\,|\,\eta,\alpha\right)=\sum_{k=0}^{k_{\max}(x,N,\gamma)}\mathfrak{p}_k^{(N)}(\eta,\alpha)\sim e_{q=1+\frac{1}{\eta\alpha}}^{-N\,r(k_{\max}\,|\,\eta,\alpha)}$$

Conclusions

- We analyzed a set of generalized binomial distributions, $\mathfrak{p}_k^{(N)}(\eta;\alpha)$, that can be interpreted as the probability of having k wins (probability η) and N-k losses (probability 1-η) in a sequence of *correlated* N trials. They preserve the symmetry win-loss, and fulfill the Leibniz triangle rule ($q_{\text{entropy}} = 1$).

- Numerical results strongly suggest that increasing N and for fixed values of $\alpha = \alpha_0$, the probability distribution function of the ratio of wins per trial evolves as a q-Gaussian, and $q(\alpha_0, N)$ increases with N.

- Numerical results strongly suggest that the attractors $(N \to \infty)$ of the probability distribution function of the ratio of wins per trial are q-Gaussians $(q<1)$. The value of $q^{att}(\alpha) \equiv \lim_{N\to\infty} q(\alpha, N)$ is analytically obtained as a function of α as $q^{att}(\alpha) = 1 - \frac{2}{\alpha - 2}$, recovering the binomial distribution limit when $\alpha \to \infty$

- The generalized binomial distributions represent a superdiffusion process.

- The generalized binomial distributions do not satisfy the Weak Law of large numbers.

- Large-deviation-like properties are satisfied by $P(N; \frac{k}{N^\gamma} < x \mid \eta, \alpha)$ distributions $(\gamma < 1)$. In particular:
  - Probability deviation q-exponentially decays.
  - The value of q is obtained as a function of the parameters of the model (η, α): $q(\eta, \alpha) = 1 + \frac{1}{\eta\alpha}$
  - Non-trivial dependence of the rate of q-exponential decay, $r = r(x, N, \gamma \mid \eta, \alpha)$, is analyzed.

- The *q-Gaussian* attractor $q^{att}(\alpha)$, and the q-exponential parameter of $\eta = 1/2$ models $q(\eta, \alpha)$ $(\forall \alpha)$, satisfy

$$2 - q^{att}(\alpha) = \frac{1}{2 - q(\alpha)\big|_{\eta=1/2}} \Leftrightarrow \frac{1}{q(\alpha)\big|_{\eta=1/2} - 1} + \frac{1}{q^{att}(\alpha) - 1} = 1 \Leftrightarrow q(\alpha)\big|_{\eta=1/2} = 2 + \frac{1}{q^{att}(\alpha) - 2}$$

- Large-deviation-like properties of $P(N; \frac{k}{N^\gamma} < x \mid \eta, \alpha)$ distributions $(\gamma > 1)$ are being studied, and so is the behavior of deviation probabilities form the expectation value ($\gamma > 1$ and $\gamma < 1$).